JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **37**, No. 2, May 2024
http://dx.doi.org/10.14403/jcms.2024.37.2.87

APPROXIMATION OF ALMOST EULER–LAGRANGE QUADRATIC MAPPINGS BY QUADRATIC MAPPINGS

John Michael Rassias, Hark-Mahn Kim† and Eunyoung Son

Abstract. For any fixed integers k, l with $kl(l-1) \neq 0$, we establish the generalized Hyers–Ulam stability of an Euler–Lagrange quadratic functional equation

$$\begin{aligned} &f(kx+ly) + f(kx-ly) + 2(l-1)[k^2 f(x) - lf(y)] \\ &\quad = l[f(kx+y) + f(kx-y)] \end{aligned}$$

in normed spaces and in non-Archimedean spaces, respectively.

1. Introduction

In 1940, S.M. Ulam [22] at the Mathematics Club of the University of Wisconsin has presented the question concerning the stability of group homomorphisms: when a solution of an equation of group homomorphism, differing slightly from a given one, must be near to the exact solution of the given equation. In 1941, Hyers [9] gave an affirmative answer to Ulam's problem for the case of approximate additive mappings on Banach spaces. In 1950, Aoki [1] has extended the Hyers–Ulam stability theorem for unbounded controlled functions. This stability result for approximate additive mappings has been further generalized and rediscovered by Rassias [19] in 1978 and by P. Găvruta [7] in 1994.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called the quadratic functional equation. Every solution of equation (1.1) is said to be a quadratic mapping. The Hyers–Ulam stability theorem for the quadratic functional equation

Received March 05, 2024; Accepted May 08, 2024.
2020 Mathematics Subject Classification: 39B72, 39B82.
Key words and phrases: Euler–Lagrange quadratic mappings, Hyers–Ulam stability, non-Archimedean normed spaces.
* This work was supported by research fund of Chungnam National University.
† Correspondence should be addressed to Hark-Mahn Kim, hmkim@cnu.ac.kr.

(1.1) has been established by Skof [21] for mappings $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. The result of Skof has been further generalized by Czerwik [4], Rassias [20], Borelli and Forti [2]. During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam stability of several functional equations, and there are many interesting results concerning these stability theorems of several functional equations [5, 8, 10, 12, 13].

In particular, Rassias investigated the Hyers–Ulam stability for the relative Euler–Lagrange functional equation

$$f(ax + by) + f(bx - ay) = (a^2 + b^2)[f(x) + f(y)] \tag{1.2}$$

in the references [16, 17, 18].

For any fixed integers k with $k \neq 0, 1$, Kim et al. [14] investigated the generalized Hyers–Ulam stability of the Euler–Lagrange quadratic functional equation

$$\begin{aligned} f(kx + y) + f(kx - y) \\ = k[f(x + y) + f(x - y)] + 2(k - 1)[kf(x) - f(y)] \end{aligned} \tag{1.3}$$

in normed spaces and in non-Archimedean normed spaces. In addition, the authors [15] have established the generalized Hyers–Ulam stability of the Euler–Lagrange quadratic functional equation

$$\begin{aligned} f(kx + ly) + f(kx - ly) \\ = kl[f(x + y) + f(x - y)] + 2(k - l)[kf(x) - lf(y)] \end{aligned} \tag{1.4}$$

in fuzzy Banach spaces, where k, l are nonzero rational numbers with $k \neq l$. Combining the equation (1.3) with (1.4), we arrive at the following functional equation

$$\begin{aligned} f(kx + ly) + f(kx - ly) + 2(l - 1)[k^2 f(x) - lf(y)] \\ = l[f(kx + y) + f(kx - y)], \end{aligned} \tag{1.5}$$

and the authors [3] recently have established the generalized Hyers–Ulam stability of the equation in fuzzy Banach spaces. In this paper, we are going to investigate the generalized Hyers–Ulam stability of the equation (1.5) in normed spaces and in non-Archimedean normed spaces for any fixed nonzero integers k, l with $l \neq 1$.

2. Stability of (1.5) in Banach spaces

In this section, let X be a normed space and Y a Banach space. For notational convenience, we would like to define an operator $D_{k,l}f(x,y)$ as

$$\begin{aligned} D_{k,l}f(x,y) \quad := \quad & f(kx+ly)+f(kx-ly)+2(l-1)[k^2f(x)-lf(y)] \\ & -l[f(kx+y)+f(kx-y)] \end{aligned}$$

for all $x, y \in X$, where k, l are fixed nonzero integer numbers with $kl(l-1) \neq 0$. Before taking up the main subject, we remark that a mapping $f : X \to Y$ between linear spaces satisfies the Euler–Lagrange functional equation (1.5) if and only if it satisfies the equation (1.1), and so, f is quadratic [3]. Now, we introduce a stability theorem for an approximate Euler–Lagrange quadratic mapping of the equation (1.5).

Theorem 2.1. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{k,l}f(x,y)\| \le \psi(x,y), \;\; x,y \in X, \tag{2.1}$$

and the perturbing function $\psi : X^2 \to [0,\infty)$ satisfies

$$\sum_{i=0}^{\infty} \frac{\psi(k^ix, k^iy)}{k^{2i}} < \infty, \;\; x,y \in X. \tag{2.2}$$

Then there exists a unique quadratic mapping $Q_1 : X \to Y$, defined by

$$Q_1(x) = \lim_{n\to\infty} \frac{f(k^nx)}{k^{2n}}, \;\; x \in X, \tag{2.3}$$

which satisfies the approximation

$$\|f(x) - Q_1(x)\| \le \frac{1}{2k^2|l-1|} \sum_{i=0}^{\infty} \frac{\psi(k^ix, 0)}{k^{2i}}, \quad x \in X. \tag{2.4}$$

Proof. Putting $y := 0$ in (2.1) and dividing by $2k^2|l-1|$, we obtain

$$\left\| \frac{f(kx)}{k^2} - f(x) \right\| \le \frac{\psi(x,0)}{2k^2|l-1|} \tag{2.5}$$

for all $x \in X$. Using the induction argument on the positive integers n, we may obtain

$$\left\| f(x) - \frac{f(k^nx)}{k^{2n}} \right\| \le \frac{1}{2k^2|l-1|} \sum_{i=0}^{n-1} \frac{\psi(k^ix, 0)}{k^{2i}}, \;\; x \in X. \tag{2.6}$$

Now, it follows from (2.6) that for any positive integers $m > n > 0$,

$$\begin{aligned}\left\|\frac{f(k^m x)}{k^{2m}} - \frac{f(k^n x)}{k^{2n}}\right\| &= \left\|\frac{f(k^{m-n+n}x)}{k^{2(m-n+n)}} - \frac{f(k^n x)}{k^{2n}}\right\| \\ &= \frac{1}{k^{2n}}\left\|f(k^n x) - \frac{f(k^{m-n}k^n x)}{k^{2(m-n)}}\right\| \\ &\le \frac{1}{2k^2|l-1|}\sum_{i=0}^{m-n-1}\frac{\psi(k^{i+n}x,0)}{k^{2(i+n)}} \qquad (2.7)\end{aligned}$$

for all $x \in X$. Since the right-hand side of the inequality (2.7) tends to 0 as $n \to \infty$, a sequence $\{\frac{f(k^n x)}{k^{2n}}\}$ is Cauchy in the Banach space Y. Therefore, we may define a mapping $Q_1 : X \to Y$ as

$$Q_1(x) = \lim_{n\to\infty}\frac{f(k^n x)}{k^{2n}}, \ x \in X.$$

Letting $n \to \infty$ in (2.6), we lead to the approximation (2.4).

Replacing (x, y) by $(k^n x, k^n y)$ in (2.1) and dividing by k^{2n}, we obtain

$$\frac{1}{k^{2n}}\left\|D_{k,l}f(k^n x, k^n y)\right\| \le \frac{1}{k^{2n}}\psi(k^n x, k^n y), \ x, y \in X.$$

Taking the limit as $n \to \infty$, we see from (2.2) and (2.3) that the mapping Q_1 satisfies the equation (1.5) and so it is quadratic.

To prove the uniqueness of quadratic mapping Q_1 satisfying the approximation (2.4), let us assume that there exists a quadratic mapping $Q_1' : X \to Y$ which satisfies the estimation (2.4). Then, we have $Q_1(k^n x) = k^{2n}Q_1(x)$ and $Q_1'(k^n x) = k^{2n}Q_1'(x)$ for all $x \in X$ and all $n \in \mathbf{N}$ because they are quadratic mappings. Hence, it follows from (2.4) that

$$\begin{aligned}\|Q_1(x) - Q_1'(x)\| &= \frac{1}{k^{2n}}\|Q_1(k^n x) - Q_1'(k^n x)\| \\ &\le \frac{1}{k^{2n}}\Big[\|Q_1(k^n x) - f(k^n x)\| + \|f(k^n x) - Q_1'(k^n x)\|\Big] \\ &\le \frac{1}{k^2|l-1|}\sum_{i=n}^{\infty}\frac{\psi(k^i x, 0)}{k^{2i}}, \quad \forall n \in \mathbf{N},\end{aligned}$$

which tends to zero as $n \to \infty$. This completes the proof. □

The following theorem is an alternative stability result concerning the stability of functional equation (1.5) controlled by the perturbing function ψ.

Theorem 2.2. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (1.5) and the perturbing function $\psi : X^2 \to [0, \infty)$ satisfies the following condition*

$$\sum_{i=1}^{\infty} k^{2i}\psi\left(\frac{x}{k^i}, \frac{y}{k^i}\right) < \infty, \ x, y \in X. \tag{2.8}$$

Then there exists a unique quadratic mapping $Q_2 : X \to Y$, defined as

$$Q_2(x) = \lim_{n\to\infty} k^{2n} f(\frac{x}{k^n}), \ x \in X,$$

which satisfies the estimation

$$\|f(x) - Q_2(x)\| \leq \frac{1}{2k^2|l-1|} \sum_{i=1}^{\infty} k^{2i}\psi(\frac{x}{k^i}, 0), \quad x \in X. \tag{2.9}$$

Proof. It follows from (2.5) that

$$\Big\|f(x) - k^2 f(\frac{x}{k})\Big\| \leq \frac{1}{2k^2|l-1|} k^2\psi(\frac{x}{k}, 0),$$

which yields the following functional inequality

$$\Big\|f(x) - k^{2n} f(\frac{x}{k^n})\Big\| \leq \frac{1}{2k^2|l-1|} \sum_{j=1}^{n} k^{2j}\psi(\frac{x}{k^j}, 0)$$

for all $x \in X$. The remaining assertions go similarly through the proof of Theorem 2.1, and thus we omit the proof. □

Now, we obtain a corollary of Theorem 2.1 in the complete normed space $(Y, \|\cdot\|)$ under the uniformly bounded condition of perturbing term $D_{k,l}f(x, y)$.

Corollary 2.3. *Let ε be a nonnegative real number and $|k| > 1$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{k,l}f(x, y)\| \leq \varepsilon, \quad x, y \in X,$$

then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{\varepsilon}{2|l-1||k^2-1|}, \quad x \in X.$$

3. Stability of (1.5) in non-Archimedean spaces

Let $\mathbf{K}$ be a field equipped with a function, so called, non-Archimedean valuation $|\cdot|_v$ from $\mathbf{K}$ into $[0,\infty)$ satisfying the following conditions:

(a) $|r|_v = 0$ if and only if $r = 0$;

(b) $|rs|_v = |r|_v|s|_v$;

(c) the strong triangle inequality, namely, $|r+s|_v \le \max\{|r|_v, |s|_v\}$ for all $r, s \in \mathbf{K}$. In this case, it is said that the pair $(\mathbf{K}, |\cdot|_v)$ is a non-Archimedean field. Then, it is clear that $|1|_v = 1 = |-1|_v$ and $|n|_v \le 1$ for all integers n.

Now, let Y be a vector space over the non-Archimedean field $\mathbf{K}$ with a non-trivial non-Archimedean valuation $|\cdot|_v$. Then a function $\|\cdot\|_v : Y \to [0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(a) $\|x\|_v = 0$ if and only if $x = 0$;

(b) $\|rx\|_v = |r|_v\|x\|_v$ for all $x \in Y$ and all $r \in \mathbf{K}$;

(c) the strong triangle inequality, namely,

$$\|x+y\|_v \le \max\{\|x\|_v, \|y\|_v\}$$

for all $x, y \in Y$. In this case, the pair $(Y, \|\cdot\|_v)$ is called a non-Archimedean normed space, and we mean that a non-Archimedean normed space $(Y, \|\cdot\|_v)$ is complete if and only if every Cauchy sequence in Y is convergent in the space Y by the norm $\|\cdot\|_v$. It follows from the strong triangle inequality that

$$\|x_n - x_m\|_v \le \max\{\|x_{j+1} - x_j\|_v : m \le j < n-1\}$$

for all $x_n, x_m \in Y$ and all $m, n \in \mathbf{N}$ with $n > m$. Therefore, a sequence $\{x_n\}$ is a Cauchy sequence in non-Archimedean normed space $(Y, \|\cdot\|_v)$ if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero in the space $(Y, \|\cdot\|_v)$. Now, we will investigate the generalized the Hyers–Ulam stability problem for the functional equation (1.5) in a complete non-Archimedean normed space Y. In this section, let X be a vector space and Y a complete non-Archimedean normed space.

Theorem 3.1. *Let $\psi : X^2 \to [0,\infty)$ be a function such that*

$$\Psi_1(x) := \lim_{n\to\infty} \max\left\{\frac{\psi(k^i x, 0)}{|k|_v^{2i}} : 0 \le i < n\right\} < \infty, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{\psi(k^n x, k^n y)}{|k|_v^{2n}} = 0$$

for all $x, y \in X$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_{k,l}f(x,y)\|_v \leq \psi(x,y), \quad x, y \in X, \tag{3.2}$$

then there exists a quadratic mapping $Q_1 : X \to Y$, defined as

$$Q_1(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}}, \quad x \in X, \tag{3.3}$$

which satisfies the equation (1.5) and the approximation

$$\|f(x) - Q_1(x)\|_v \leq \frac{1}{|2|_v|l-1|_v|k|_v^2} \Psi_1(x) \quad x \in X. \tag{3.4}$$

Moreover, if

$$\lim_{m\to\infty} \frac{\Psi_1(k^m x)}{|k|_v^{2m}} = \lim_{m\to\infty} \lim_{n\to\infty} \max \left\{ \frac{\psi(k^j x, 0)}{|k|_v^{2j}} : m \leq j < m+n \right\} = 0$$

for all $x \in X$, then Q_1 is a unique quadratic mapping satisfying (3.4).

Proof. Putting $y := 0$ in (3.2) and dividing by $|2|_v|l-1|_v|k|_v^2$, we arrive at

$$\|\frac{f(kx)}{k^2} - f(x)\|_v \leq \frac{\psi(x,0)}{|2|_v|l-1|_v|k|_v^2} \tag{3.5}$$

for all $x \in X$, where $|k|_v \leq 1$ is a non-Archimedean valuation. Replacing x by $k^n x$ in (3.5) and dividing by $|k|_v^{2n}$,

$$\left\| \frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^n x)}{k^{2n}} \right\|_v \leq \frac{\psi(k^n x, 0)}{|2|_v|l-1|_v|k|_v^{2n+2}}, \quad x \in X. \tag{3.6}$$

Since the right-hand side of the inequality (3.6) tends to 0 as $n \to \infty$, a sequence $\{\frac{f(k^n x)}{k^{2n}}\}$ is Cauchy in the complete non-Archimedean space $(Y, \|\cdot\|_v)$. Therefore, we may define a mapping $Q_1 : X \to Y$ as

$$Q_1(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{2n}}, \quad x \in X.$$

Using the induction argument and the strong triangle inequality, we may figure out

$$\left\| f(x) - \frac{f(k^n x)}{k^{2n}} \right\|_v \leq \frac{1}{|2|_v|l-1|_v|k|_v^2} \max \left\{ \frac{\psi(k^i x, 0)}{|k|_v^{2i}} : 0 \leq i < n \right\}$$

for all $x \in X$. Letting $n \to \infty$ in the last inequality, we lead to the approximation (3.4).

Next, we have to show that the mapping Q_1 defined above satisfies equation (1.5). Replacing (x,y) by $(k^n x, k^n y)$ in (3.2), and then dividing the resulting inequality by $|k|_v^{2n}$, it follows that

$$\frac{1}{|k|_v^{2n}}\|D_{k,l}f(k^n x, k^n y)\|_v \le \frac{1}{|k|_v^{2n}}\psi(k^n x, k^n y), \quad x, y \in X.$$

Taking the limit as $n \to \infty$, it follows from (3.1) and (3.3) that

$$D_{k,l}Q_1(x,y) = 0, \quad x, y \in X.$$

Therefore, the mapping Q_1 satisfies the equation (1.5) and so it is quadratic.

In the last, we now prove the uniqueness of the quadratic mapping Q_1 satisfying the inequality (3.4). Let us assume that there exists a quadratic mapping $Q_1' : X \to Y$ which satisfies the inequality (3.4). Then, we have $Q_1(k^m x) = k^{2m}Q_1(x)$ and $Q_1'(k^m x) = k^{2m}Q_1'(x)$ for all $x \in X$ and all $m \in \mathbf{N}$. Hence, it follows from (3.4) that for all $x \in X$

$$\begin{aligned}
\|Q_1(x) \quad - \quad Q_1'(x)\|_v &= \frac{1}{|k|_v^{2m}}\|Q_1(k^m x) - Q_1'(k^m x)\|_v \\
&\le \frac{1}{|k|_v^{2m}} \max\left\{\|Q_1(k^m x) - f(k^m x)\|_v, \|f(k^m x) - Q_1'(k^m x)\|_v\right\} \\
&\le \frac{1}{|2|_v|l-1|_v|k|_v^2} \lim_{n\to\infty} \max\left\{\frac{\psi(k^{m+i}x, 0)}{|k|_v^{2(m+i)}} : 0 \le i < n\right\} \\
&= \frac{1}{|2|_v|l-1|_v|k|_v^2} \lim_{n\to\infty} \max\left\{\frac{\psi(k^j x, 0)}{|k|_v^{2j}} : m \le j < m+n\right\} \\
&= \frac{1}{|2|_v|l-1|_v|k|_v^2} \frac{\Psi_1(k^m x)}{|k|_v^{2m}}, \quad \forall\, m \in \mathbf{N},
\end{aligned}$$

which tends to zero as $m \to \infty$. This completes the proof. □

The following is an alternative stability theorem of Theorem 3.1 in the complete non-Archimedean normed space $(Y, \|\cdot\|_v)$.

Theorem 3.2. *Let $\psi : X^2 \to [0,\infty)$ be a function such that*

$$\begin{aligned}
\Psi_2(x) := \lim_{n\to\infty} \max\left\{|k|_v^{2i}\psi(\frac{x}{k^i}, 0) : 1 \le i \le n\right\} &< \infty \\
\lim_{n\to\infty} |k|_v^{2n}\psi(\frac{x}{k^n}, \frac{y}{k^n}) &= 0
\end{aligned} \tag{3.7}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.2) for all $x, y \in X$, then there exists a quadratic mapping

$Q_2 : X \to Y$, defined by

$$Q_2(x) = \lim_{n\to\infty} k^{2n} f(\frac{x}{k^n}), \quad x \in X, \tag{3.8}$$

which satisfies the equation (1.5) and the approximation

$$\|f(x) - Q_2(x)\|_v \le \frac{1}{|2|_v|l-1|_v|k|_v^2}\Psi_2(x), \quad x \in X. \tag{3.9}$$

Moreover, if $\lim_{l\to\infty} |k|_v^{2l}\Psi_2(\frac{x}{k^l}) = 0$ for all $x \in X$, then Q_2 is a unique quadratic mapping satisfying (3.9).

Proof. Noting the inequality (3.5), we figure out

$$\left\|f(x) - k^2 f\left(\frac{x}{k}\right)\right\|_v \le \frac{1}{|2|_v|l-1|_v|k|_v^2}|k|_v^2\psi\left(\frac{x}{k},0\right), x \in X. \tag{3.10}$$

Replacing x by $\frac{x}{k^{n-1}}$ in (3.10) and multiplying it by $|k|_v^{2(n-1)}$, we have

$$\left\|k^{2(n-1)} f\left(\frac{x}{k^{n-1}}\right) - k^{2n} f\left(\frac{x}{k^n}\right)\right\|_v \le \frac{1}{|2|_v|l-1|_v|k|_v^2}|k|_v^{2n}\psi\left(\frac{x}{k^n},0\right)$$

for all $x \in X$. Since the right-hand side in the last inequality tends to 0 as $n \to \infty$, the sequence $\{k^{2n} f(\frac{x}{k^n})\}$ is Cauchy in the complete non-Archimedean space $(Y, \|\cdot\|_v)$. Therefore, one can define a mapping $Q_2 : X \to Y$ by

$$Q_2(x) = \lim_{n\to\infty} k^{2n} f(\frac{x}{k^n}), \quad x \in X.$$

Using induction on positive integers n, one obtains that

$$\|f(x) - k^{2n} f(\frac{x}{k^n})\|_v \le \frac{1}{|2|_v|l-1|_v|k|_v^2} \max\left\{|k|_v^{2i}\psi(\frac{x}{k^i},0) : 1 \le i \le n\right\}$$

for all $x \in X$. Letting $n \to \infty$ in the last inequality, we arrive at the approximation (3.9) near f.

The remaining assertions are similar to those of Theorem 3.1. □

As a corollary of Theorem 3.2, we obtain the following stability result in the complete non-Archimedean normed space $(Y, \|\cdot\|_v)$ under the uniformly bounded condition of perturbing term $D_{k,l}f(x,y)$.

Corollary 3.3. *Let ε be a nonnegative real number and $|k|_v < 1$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_{k,l}f(x,y)\|_v \le \varepsilon, \quad x, y \in X,$$

then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.5) and the approximation

$$\|f(x) - Q(x)\|_v \leq \frac{\varepsilon}{|2|_v |l-1|_v}, \quad x \in X.$$

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Jpn., **2** (1950), pp. 64–66.

[2] C. Borelli, G.L. Forti, *On a general Hyers–Ulam stability,* Internat. J. Math. Math. Sci., **18**, (1995), pp. 229–236.

[3] I.-S. Chang, H.-M. Kim, John M. Rassias, *Approximate Euler–Lagrange quadratic mappings in fuzzy Banach spaces,* J. Comput. Anal. Appl., **32** (1) (2024), pp. 103–116.

[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, **62** (1992), pp. 59–64.

[5] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[6] G.L. Forti, *Hyers–Ulam stability of functional equations in several variables*, Aequationes Math., **50**, (1995), pp. 143–190.

[7] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**, (1994), pp. 431–436.

[8] M. Eshaghi Gordji, H. Khodaei, H.-M. Kim, *Approximate quartic and quadratic mappings in quasi-Banach spaces,* Internat. J. Math. Math. Sci., **2011** (2011), Article ID 734567, 18 pages.

[9] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl Acad. Sci., **27** (1941), pp. 222–224.

[10] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[11] K.-W. Jun, Y.-H. Lee, *On the Hyers–Ulam–Rassias stability of a pexiderized quadratic inequality*, Math. Inequal. Appl., **4**(1) (2001), pp. 93–118.

[12] S.-M. Jung, *On the Hyers–Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl., **222** (1998), pp. 126–137.

[13] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, Springer optimization and its applications, **48**, Springer,New York, 2011.

[14] H.-M. Kim, J. Lee, E. Son, *Approximate Euler–Lagrange quadratic mappings,* J. Inequal. Appl., **2012**, 2012:58, 14 pages.

[15] H.-M. Kim, John M. Rassias, J. Lee, *Fuzzy Approximation of Euler–Lagrange quadratic mappings,* J. Inequal. Appl., **2013**, 2013:358, 15 pages.

[16] J.M. Rassias, *On the stability of the Euler–Lagrange functional equation*, Chinese J. Math., **20** (1992), pp. 185–190.

[17] J.M. Rassias, *On the stability of the non-linear Euler–Lagrange functional equation in real normed linear spaces*, J. Math. Phys. Sci., **28** (1994), pp. 231–235.

[18] J.M. Rassias, *On the stability of the general Euler–Lagrange functional equation*, Demonstratio Math., **29** (1996), pp. 755–766.

[19] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), pp. 297–300.
[20] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251** (2000), pp. 264–284.
[21] F. Skof, *Local properties and approximations of operators(Italian)*, Rend. Sem. Mat. Fis. Milano, **53** (1983), pp. 113–129.
[22] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Publication, New York, 1960.

John Michael Rassias
Pedagogical Department E.E.
National and Capodistrian University of Athens
Athens, 15342, Greece
E-mail: `jrassias@primedu.uoa.gr`

Hark-Mahn Kim
Department of Mathematics
Chungnam National University
Daejeon 34134, Korea
E-mail: `hmkim@cnu.ac.kr`

Eunyoung Son
Department of Mathematics
Chungnam National University
Daejeon 34134, Korea
E-mail: `sey8405@hanmail.net`